BICOMMUTANTS AND RANGES OF DERIVATIONS

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Abstract. Let V be a vector space, V^* its dual space and $\mathrm{L}(V)$ the algebra of all linear operators on V. For a subset R of $\mathrm{L}(V)$ let R'' be its bicommutant and $\tilde{R}$ the set of all adjoint operators a^*, $a \in R$. If R is a left noetherian subalgebra of $\mathrm{L}(V)$, then $(\tilde{R})'' = \widetilde{R''}$. When R is singly generated R'' is described precisely.

Further, for any two operators $a, b \in \mathrm{L}(V)$, $b \in (a)''$ if and only if the derivations d_a and d_b satisfy $d_b(\mathrm{F}(V)) \subseteq d_a(\mathrm{F}(V)))$, where $\mathrm{F}(V)$ is the set of all finite rank operators on V. In this case the inclusion $d_b(\mathrm{L}(V)) \subseteq d_a(\mathrm{L}(V))$ also holds.

1. Introduction

For a subset R of the algebra $\mathrm{L}(V)$ of all linear operators on a vector space V let R' be its commutant (= the set of all operators in $\mathrm{L}(V)$ that commute with all elements of R) and $R'' = (R')'$ its bicommutant. We denote by $(a)'$ and $(a)''$ the commutant and the bicommutant of a single operator $a \in \mathrm{L}(V)$. As usual, a^* denotes the adjoint of a, acting on the dual space V^*.

If $b \in \mathrm{L}(V)$ is such that $b^* \in (a^*)''$, then b^* commutes with all operators $e \in (a^*)'$, hence in particular with all operators of the form $e = c^*$, where $c \in (a)'$. Then $bc = cb$, hence $b \in (a)''$. This proves that $b^* \in (a^*)''$ implies that $b \in (a)''$. Is the reverse of this implication also true? That is:

Does $b \in (a)''$ imply that $b^* \in (a^*)''$?

To be in $(a^*)''$ the operator b^* must commute with each $e \in (a^*)'$, but not all such e are adjoints of operators on V. So the question is nontrivial; the answer to the analogous question in the context of bounded operators on Banach spaces can be negative (Section 5). Therefore it is perhaps surprising that the answer for general linear operators is positive.

An operator a on a vector space V over an arbitrary field $\mathbb{F}$ introduces to V the structure of a module over the principal ideal domain $R = \mathbb{F}[t]$ of polynomials through the correspondence $t \mapsto a$. We will study the above question for left noetherian subalgebras R of $\mathrm{L}(V)$ instead of just singly generated ones, this will not introduce any additional difficulties. (An algebra R is left noetherian if every left ideal of R is finitely generated, see [16] or [20].) Observe that R' is just the algebra $\mathrm{L}_R(V)$ of all R-module endomorphisms of V, and that V^* is a right R-module by $\langle \rho r, \xi \rangle := \langle \rho, r\xi \rangle$, where $\rho \in V^*$, $\xi \in V$ and $r \in R$. (Here we are using the convenient notation $\langle \rho, \xi \rangle$ for the value of a functional ρ at the vector ξ.) To prove the affirmative answer to the above question, we will first show that for any R modules U and V the right R-module homomorphisms $g \in \mathrm{L}(V^*, U^*)_R$ can

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be interpolated on finite subsets of V^* and U by adjoints of maps $f \in \mathrm{L}_R(U,V)$ (Theorem 2.2). Then we will also show that for a left noetherian algebra $R \subseteq \mathrm{L}(V)$ each map c in the bicommutant of the set $\{r^* : r \in R\}$ is of the form $c = b^*$ for some $b \in R''$ (Theorem 2.4). This means in particular that $\mathrm{L}(V^*)_R$-module homomorphisms of V^* are automatically continuous in the weak* topology of V^*. This results can not be extended to the context of general bounded operators on Banach spaces. (However, under some specific situations, there are instances of automatic weak* continuity, for example, the work of Hofmeier and Wittstock [8] concerning certain maps on $B(H)$, or the automatic weak*-continuity of multipliers on dual operator spaces [2].)

In the case $R = \mathbb{F}[a]$ is the polynomial ring in an operator $a \in \mathrm{L}(V)$ the above mentioned results enable us to give a precise description of the center $(a)''$ of the endomorphism ring $\mathrm{L}_R(V)$ (Theorem 3.9). It turns out that if V is torsion-free, then $(a)''$ is isomorphic to a subalgebra of the algebra $\mathbb{F}(t)$ of rational functions. If the torsion submodule of V is not 0, but V is not torsion, then each central element b of $\mathrm{L}_R(V)$ induces a decomposition $V = T_b \oplus W_b$, such that b acts on W_b as the multiplication by a rational function of a, and T_b is a finite direct sum of torsion submodules of bounded orders, on each of which b acts as a polynomial in a. For torsion modules the center of $\mathrm{L}_R(V)$ is already known if R is a general principal ideal ring (see [13, p. 72] or [15] in the case $R = \mathbb{Z}$).

Our initial motivation for studying the above question was the range inclusion problem for derivation ranges. If a is an element of an algebra A, the *derivation* induced by a is the map d_a on A, defined by $d_a(x) = ax - xa$. The kernel of d_a is just the commutant of a in A, but the range of d_a also turns out to be interesting. If b is another element of A, we may ask, when is the range of d_b contained in the range of d_a. This problem was studied in the past by several authors, especially in the case when A is the algebra of all bounded operators on a Hilbert space H. Very interesting results were obtained by Johnson and Williams [11] in the case a is a normal operator, and their work was continued for example by Fong [6], Kissin and Shulman [14], Brešar [3] and in [4]. Some of their results are of such a nature that one would expect them to hold for much larger class of operators a than just normal ones. But when trying to show this in a complete generality we encountered certain analytic difficulties. We found, however, that the problem is interesting also in the purely algebraic context and, since the methods in this case are completely different from those required for bounded operators, we decided to study this case separately. We will see (Theorem 4.1) that for linear operators a and b on a vector space V the condition $b \in (a)''$ is equivalent to $d_b(\mathrm{F}(V)) \subseteq d_a(\mathrm{F}(V))$, where $\mathrm{F}(V)$ is the ideal in $\mathrm{L}(V)$ of finite rank operators. This condition always implies that $d_{b^*}(\mathrm{L}(V^*)) \subseteq d_{a^*}(\mathrm{L}(V^*))$. Moreover (Theorem 4.3), it also implies that $d_b(\mathrm{L}(V)) \subseteq d_a(\mathrm{L}(V))$.

2. Bicommutants and adjoints

2.1. **Approximation of operators on V^* by adjoints of operators on V.** As usual, regard any vector space V as a subspace in its bidual V^{**} through the natural map $V \to V^{**}$.

Lemma 2.1. *For each $a \in \mathrm{L}(V)$ every element $\theta \in \ker a^{**}$ can be approximated by elements from* $\ker a$ *in the following sense: for each finite subset $\{\rho_j : j = 1, \dots, n\}$ of V^* there exist $\xi \in \ker a$ such that $\langle \rho_j, \xi \rangle = \langle \rho_j, \theta \rangle$ for all j.*

More generally, if R is a subalgebra of $\mathrm{L}(V)$*,* J *a finitely generated left ideal of* R*,* $\mathrm{ann}_V(J)$ *the annihilator of* J *in* V*, and* $\mathrm{ann}_{V^{**}}(J)$ *the annihilator of* J *in* V^* *(where* R *acts on* V^{**} *via second adjoints of elements of* R*), then elements of* $\mathrm{ann}_{V^{**}}(J)$ *can be approximated by elements of* $\mathrm{ann}_V(J)$ *in the above sense.*

Proof. It is well-known (and elementary) that for every $a \in \mathrm{L}(V)$ the equality

$$\ker a^* = (\mathrm{im}\, a)^{\perp} \tag{2.1}$$

holds, where $(\mathrm{im}\, a)^{\perp}$ is the annihilator of $\mathrm{im}\, a$ in V^*. Similarly

$$\mathrm{im}\, a^* = (\ker a)^{\perp}. \tag{2.2}$$

(For a proof of the nontrivial inclusion $(\ker a)^{\perp} \subseteq \mathrm{im}\, a^*$, note that for each $\rho \in (\ker a)^{\perp}$ the map $a\xi \mapsto \rho(\xi)$ is well defined on $\mathrm{im}\, a$, and any of its linear extensions $\omega \in V^*$ satisfies $a^*(\omega) = \omega \circ a = \rho$.) Thus

$$\ker a^{**} = (\mathrm{im}\, a^*)^{\perp} = (\ker a)^{\perp\perp}.$$

Since for each subspace U of V, $U^{\perp\perp}$ is naturally isomorphic to the bidual U^{**} of U, we infer that $\ker a^{**} = (\ker a)^{**}$ (where '=' means the natural isomorphism), so the first statement of the lemma reduces to the well-known density of U in U^{**}.

To prove the second statement of the lemma, let $\{a_1, \ldots, a_m\}$ be a set of generators of J and denote by b the operator $(a_1, \ldots, a_m) : V \to V^m$. Since

$$\mathrm{ann}_V(J) = \cap_{a\in J} \ker a = \cap_{i=1}^m \ker a_i = \ker b \text{ and } \mathrm{ann}_{V^{**}}(J) = \cap_{i=1}^m \ker a_i^{**} = \ker b^{**},$$

the statement follows from the argument of the previous paragraph. □

Theorem 2.2. *Let* R *be a left noetherian unital algebra over any field* $\mathbb{F}$ *and* U*,* V *any left* R*-modules. Each* $g \in \mathrm{L}(V^*, U^*)_R$ *can be approximated by adjoints of elements of* $\mathrm{L}_R(U, V)$ *in the following sense: for every finite subsets* G *of* U *and* H *of* V^* *there exists* $f \in \mathrm{L}_R(U, V)$ *such that*

$$\langle g(\rho), \xi\rangle = \langle \rho, f(\xi)\rangle \quad \textit{for all } \rho \in H \textit{ and } \xi \in G.$$

Proof. Let us first consider the case when U is finitely generated as an R-module, hence of the form $U = R^n/A$ for some $n \in \mathbb{N}$ and a left submodule A of R^n. Since the space $\mathrm{L}_R(R^n, V)$ can be naturally identified with V^n, we thus have a natural isomorphism

$$\mathrm{L}_R(U, V) = \{f \in \mathrm{L}_R(R^n, V) : \ f(A) = 0\} = \mathrm{ann}_{V^n}(A).$$

Under this isomorphism a map $f \in \mathrm{L}_R(U, V)$ corresponds to the element

$$(fq(e_1), \ldots, fq(e_n)) \in \mathrm{ann}_{V^n}(A),$$

where the e_j are the usual basic elements of R^n (e_j has 1 on the j-th position and 0 elsewhere) and $q : R^n \to U$ is the quotient map. Similarly, using the natural isomorphism

$$\mathrm{L}(V^*, U^*)_R = \mathrm{L}_R(U, V^{**}) \quad (g \mapsto g^*|U),$$

we have that

$$\mathrm{L}(V^*, U^*)_R = \mathrm{ann}_{(V^{**})^n}(A).$$

A is a left submodule of R^n; in order to apply Lemma 2.1, we will first convert it into a left ideal in the ring $\mathrm{M}_n(R)$ of all $n \times n$ matrices over R. Namely, regard R^n as the space $\mathrm{M}_{1,n}(R)$ of row matrices (so that in particular the elements of A

are rows) and let $J = \mathrm{M}_{n,1}(R)A$. Then J is a left ideal in $\mathrm{M}_n(R)$, and we can recapture A from J as

$$A = RA = \mathrm{M}_{1,n}(R)\mathrm{M}_{n,1}(R)A = \mathrm{M}_{1,n}(R)J.$$

This implies that an element ξ of V^n (or of $(V^{**})^n$) is annihilated by A if and only if it is annihilated by J (where V^n and $(V^{**})^n$ carry the left $\mathrm{M}_n(R)$-module structure obtained from the R-modules V and V^{**} in the usual way, regarding elements of V^n and $(V^*)^n$ as columns). Thus

$$\mathrm{ann}_{V^n}(A) = \mathrm{ann}_{V^n}(J) \text{ and } \mathrm{ann}_{(V^{**})^n}(A) = \mathrm{ann}_{(V^{**})^n}(J).$$

Since R is left noetherian, each submodule of a finitely generated left R-module is finitely generated [20], so A in particular is finitely generated, which implies that J is a finitely generated left ideal in $\mathrm{M}_n(R)$. (If $\{a_j : j = 1, \ldots, a_m\}$ is a generating set of A and $\{e_i : i = 1, \ldots, n\}$ the usual basis of $\mathrm{M}_{n,1}(R)$, then $\{e_i a_j : i = 1, \ldots, m,\ j = 1, \ldots n\}$ generates J.) Since the ring $\mathrm{M}_n(R)$ is left noetherian [20, p. 395], we can now apply Lemma 2.1: it follows that if $\theta \in (V^{**})^n = (V^n)^{**}$ is in $\mathrm{ann}_{(V^{**})^n}(A)$ then for each finite subset H_0 of $(V^*)^n$ there exists $\eta \in \mathrm{ann}_{V^n}(A)$ such that

$$\langle \omega, \theta \rangle = \langle \omega, \eta \rangle \text{ for all } \omega \in H_0.$$

We must now translate this approximation to the context of maps appearing in the lemma.

A map $g \in \mathrm{L}(V^*, U^*)_R$ corresponds under the above identification to the element $\theta := (g^*q(e_1), \ldots, g^*q(e_n))$ of $\mathrm{ann}_{(V^{**})^n}(A)$. For each element ξ of a given finite subset G of U choose a representation $\xi = \sum_{j=1}^n r_j(\xi)q(e_j)$ and denote by $r(\xi)$ the element $(r_1(\xi), \ldots, r_n(\xi))$ of R^n. Further, for each ρ in a given finite subset H of V^* let $\rho r(\xi)$ be the element of $(V^*)^n$ defined by

$$\rho r(\xi) = (\rho r_1(\xi), \ldots, \rho r_n(\xi)).$$

Choose a finite subset H_0 of $(V^*)^n$ so that H_0 contains all the elements $\rho r(\xi)$ for $\rho \in H$ and $\xi \in G$. Let $\eta \in \mathrm{ann}_{V^n}(A)$ be the approximation for θ as in the previous paragraph and let $f \in \mathrm{L}_R(U, V)$ correspond to η. Then for each $\xi \in G$ and $\rho \in H$ we have

$$\begin{aligned}\langle g(\rho), \xi \rangle = \langle \rho, g^*(\xi) \rangle &= \sum_{j=1}^n \langle \rho, r_j(\xi) g^* q(e_j) \rangle = \langle \rho r(\xi), \theta \rangle \\ &= \langle \rho r(\xi), \eta \rangle = \sum_{j=1}^n \langle \rho r_j(\xi), f q(e_j) \rangle = \langle \rho, f(\xi) \rangle.\end{aligned}$$

In general (if U is not necessarily finitely generated), U is the union of an increasing net of finitely generated submodules U_k. Then a homomorphism $f : U \to V$ determines the collection (f_k) of its restrictions $f_k = f|U_k$; conversely, any collection of homomorphisms $f_k \in \mathrm{L}_R(U_k, V)$, which are compatible in the sense that $f_k|U_l = f_l$ if $l \leq k$, defines a homomorphism from U to V. Thus

$$\mathrm{L}_R(U, V) = \lim_{\leftarrow} \mathrm{L}_R(U_k, V) := \{(f_k) \in \prod_k \mathrm{L}_R(U_k, V) : l \leq k \Rightarrow f_k|U_l = f_l\}.$$

Similarly

$$\mathrm{L}(V^*, U^*)_R = \mathrm{L}_R(U, V^{**}) = \lim_{\leftarrow} \mathrm{L}_R(U_k, V^{**}).$$

Since each finite subset G of U is contained in some U_k and each $g \in \mathrm{L}(V^*, U^*)_R$ acts on G as the restriction $g^*|U_k$, the proof is reduced to the case of finitely generated modules U_k. $\square$

The notion of the inverse limit, used at the end of the above proof, is explained in detail e.g. in [20, Section 1.8], but here we do not need very much of it beyond the notation. We note that the approximation in Theorem 2.2 is in a weaker sense than, say, the one in the classical Jacobson density theorem [9], [7, p. 159], but no assumption of semisimplicty is needed in Theorem 2.2. Now we show by an example that the left noetherian condition on R in Theorem 2.2 is not redundant.

Example 2.3. Let V be an infinite dimensional vector space and W a weak* dense subspace of V^*, different from V^*. (For example, $W = \ker \theta$ for some $\theta \in V^{**} \setminus V$.) Let $R = \mathrm{L}(V)$, $J = \{a \in R : \operatorname{im} a^* \subseteq W\}$ (so that J is a left ideal in R) and $U := R/J$. Then, denoting by $S_\perp$ and $S^\perp$ the annihilators of a subset $S \subseteq V^*$ in V and in V^{**} (respectively), and identifying $\mathrm{L}_R(R, V)$ with V, we have that

$$\mathrm{L}_R(U, V) = \operatorname{ann}_V(J) = \cap_{a\in J} \ker a = (\sum_{a\in J} \operatorname{im} a^*)_\perp = W_\perp = 0,$$

but

$$\mathrm{L}(V^*, U^*)_R = \mathrm{L}_R(U, V^{**}) = \cap_{a\in J} \ker a^{**} = (\sum_{a\in J} \operatorname{im} a^*)^\perp = W^\perp \neq 0.$$

2.2. **Bicommutants and adjoints.** Let us now study the relation between the operations of taking the bicommutant and the adjoint.

Theorem 2.4. *If R is a left noetherian subalgebra of $\mathrm{L}(V)$, then*

$$(\tilde{R})'' = \widetilde{R''}, \quad \text{where } \tilde{R} := \{a^* : a \in R\} \text{ and } \widetilde{R''} := \{b^* : b \in R''\}.$$

Proof. First we will prove the inclusion

$$\widetilde{R''} \subseteq (\tilde{R})''.$$

For each $b \in R''$ we have to show that $b^*g = gb^*$ for all $g \in \tilde{R}'$, or equivalently, that

$$\langle b^*g\rho, \xi\rangle = \langle gb^*\rho, \xi\rangle$$

for all $\xi \in V$ and $\rho \in V^*$. Given g, ξ and ρ, by Theorem 2.2 there exists $f \in R'$ such that

$$\langle g\rho, b\xi\rangle = \langle \rho, fb\xi\rangle \quad \text{and} \quad \langle gb^*\rho, \xi\rangle = \langle b^*\rho, f\xi\rangle.$$

Hence

$$\langle b^*g\rho, \xi\rangle = \langle g\rho, b\xi\rangle = \langle \rho, fb\xi\rangle = \langle \rho, bf\xi\rangle = \langle b^*\rho, f\xi\rangle = \langle gb^*\rho, \xi\rangle.$$

To prove the reverse inclusion, $\widetilde{R''} \supseteq (\tilde{R})''$, it suffices to show that each $c \in (\tilde{R})''$ is equal to b^* for some $b \in \mathrm{L}(V)$, since b is then necessarily in R''. (Indeed, $b^* = c$ commutes with all elements of $(\tilde{R})'$, in particular with all elements of the form f^*, where $f \in R'$. Hence $bf = fb$ and therefore b must be in $b \in R''$.) If we can show that V is an invariant subspace for c^*, then with $b := c|V$ we will have $c = b^*$. The condition $c^*V \subseteq V$ is clearly equivalent to the condition

$$c^{**}(V^\perp) \subseteq V^\perp,$$

where $V^\perp$ is the annihilator in $V^{***} = (V^{**})^*$ of the subspace V of V^{**}, that is $V^\perp = \{\sigma \in V^{***} : \sigma(V) = 0\}$. To prove this last condition, recall that the adjoint

of the natural inclusion $V \to V^{**}$ provides us with an idempotent $p \in \mathrm{L}(V^{***})$ with the range V^* and the kernel $V^\perp$. For each $a \in R$ the range V^* of p is invariant under a^{***} (since $a^{***}|V^* = a^*$). But the kernel $V^\perp$ of p is also invariant under a^{***}: indeed, for any $\sigma \in V^\perp$ and $\xi \in V$ we compute that $\langle a^{***}\sigma, \xi\rangle = \langle \sigma, a^{**}\xi\rangle = 0$, since $a^{**}\xi = a\xi \in V$. The fact that $\ker p$ and $\operatorname{im} p$ are both invariant under a^{***} means that p commutes with a^{***}, hence $p \in \mathrm{L}(V^{***})_R$. Note that $\tilde{R}$ is a right noetherian subalgebra of $\mathrm{L}(V^*)$. Since $c \in (\tilde{R})'' = (\mathrm{L}(V^*)_R)'$, it follows from the version for right modules of what we have already proved in the previous paragraph (applied to the right module V^*) that $c^* \in (\mathrm{L}_R(V^{**}))'$, hence by the same argument again $c^{**} \in (\mathrm{L}(V^{***})_R)'$. We conclude that c^{**} commutes with p and therefore the subspace $V^\perp = \ker p$ (and also $\operatorname{im} p$) is indeed invariant under c^{**}. □

Observe that if R is generated by a single operator a (and the identity) Theorem 2.4 says that the center of the algebra $\mathrm{L}(V^*)_R$ consists of all operators b^*, where b is in the center of $\mathrm{L}_R(V)$.

3. Bicommutants of linear operators

From now on R will denote the algebra $\mathbb{F}[t]$ of polynomials with coefficients in a field $\mathbb{F}$, and $K = \mathbb{F}(t)$ will be the field of all rational functions. A vector space V over $\mathbb{F}$ will be regarded as an R-module via $t \mapsto a$, where $a \in \mathrm{L}(V)$ will be fixed. Thus, for $p \in R$ and $\xi \in V$, $p\xi$ means $p(a)\xi$.

3.1. **Preliminaries.** Recall that an R-module V is *torsion* if each $\xi \in V$ is annihilated by some nonzero $p \in R$, that is $p\xi = 0$. Thus V is torsion means that a is *locally algebraic*. On the other hand, V is called *torsion-free* if $p\xi \neq 0$ for all nonzero $p \in R$ and $\xi \in V$.

Consider now an example that will be useful later.

Example 3.1. Let $V = \mathbb{F}^{(\mathbb{N})}$ be the vector space of all sequences with the entries in a field $\mathbb{F}$ that have only finitely many nonzero terms. Let $a \in \mathrm{L}(V)$ be the shift to the right:

$$a(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots).$$

V has a cyclic vector $\xi = (1, 0, 0, \ldots)$ for a and $p(a)\xi \neq 0$ for all nonzero polynomials $p \in R$, hence V, as a module over R, is isomorphic to R. Since R is commutative this implies that $(a)'' = (a)' \cong \mathrm{L}_R(R) = R$.

The dual R^* of R is isomorphic to V^*, hence to the module $\mathbb{F}^{\mathbb{N}}$ of all sequences with the entries in $\mathbb{F}$, where the module operation is given by the backward shift $a^*(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots)$. Observe that V^* is not a torsion module: it is possible to recursively define $\eta_n \in \mathbb{F}$ such that all the translates $(a^*)^n\eta$ $(n \in \mathbb{N})$ of the vector $\eta := (\eta_0, \eta_1, \ldots)$ are linearly independent, hence $p(a^*)\eta \neq 0$ for all nonzero polynomials $p \in \mathbb{F}[t]$. Indeed, set $\eta_0 = 1$ and assume inductively that $\eta_0, \ldots, \eta_{2n}$ have already been found such that the determinant

$$\delta_n = \begin{vmatrix} \eta_0 & \eta_1 & \ldots & \eta_n \\ \eta_1 & \eta_2 & \ldots & \eta_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ \eta_n & \eta_{n+1} & \ldots & \eta_{2n} \end{vmatrix}$$

is nonzero. Then we can find η_{2n+1}, η_{2n} in $\mathbb{F}$ such that the corresponding determinant δ_{n+1} is nonzero. (We can even choose $\eta_{2n+1} = 0$.)

We would like to describe the center Z of $\mathrm{L}_R(V)$ (which is just $(a)''$). By Theorem 2.4 Z is isomorphic to the center of $\mathrm{L}(V^*)_R$ and this will turn out to be a helpful information. A module V is called *injective* if it is a direct summand in any module in which it is a submodule [7, p. 204], [16, p. 60]. This is equivalent to the requirement that for each pair of modules $U \subseteq W$ every module homomorphism $f : U \to V$ extends to a module homomorphism $W \to V$. A module over a principal ideal domain R is injective if and only if V is *divisible* (see [20] or [16, p. 71]), which means that for every $\eta \in V$ and every nonzero $p \in R$ there exists $\xi \in V$ such that $p\xi = \eta$.

If V is torsion free, then so are all finitely generated submodules of V and, since R is a principal ideal ring, such submodules are free, hence flat [16, p. 123]. Therefore V is flat, hence its dual V^* is injective by [16, Lemma 3.5]. Consequently by [16, Theorem 3.48] V^* is a direct sum of indecomposable injective modules W_k. By [16, Example 3.63] every such W_k is isomorphic to either $K = \mathbb{F}(t)$ or to one of the primary summands $R(p^\infty)$ of the torsion module K/R, where $p \in R$ is prime.

We note that $R(p^\infty)$ is equal to the union of the increasing sequence of cyclic submodules $C_n = R(p^{-n} + R) \cong R/(p^n)$ $(n = 1, 2, \ldots)$, which are invariant under all endomorphisms of $R(p^\infty)$. Since the endomorphism algebra of the cyclic module $R/(p^n)$ is isomorphic to $R/(p^n)$ (namely, each endomorphism is determined by the image of the generator $1+(p^n)$), it follows that the endomorphism algebra of $R(p^\infty)$ can be identified with

$$\lim_{\leftarrow} \mathrm{L}_R(C_n) = \lim_{\leftarrow} R/(p^n) =: \hat{R}_p.$$

This algebra is analogous to the ring of p-adic integers [7, p. 54]; its elements can be regarded as formal power series of the form

$$f = \sum_{j=0}^{\infty} f_j p^j, \tag{3.1}$$

where $f_j \in R$ are of degree less than the degree of p. Each such series acts as an endomorphism of $R(p^\infty)$ by multiplication: $f \cdot p^{-k} = \sum_{j=0}^{k-1} f_j p^{j-k}$ $(k = 1, 2, \ldots)$. Observe that on each cyclic submodule C_n this multiplication by f has the same effect as the multiplication by the polynomial polynomial $\sum_{j=0}^{n-1} f_j p^j \in R$. Further, in this way $R/(p^n) \cong C_n$ becomes a module over $\hat{R}_p$, and an R-module homomorphism of such modules is automatically an $\hat{R}_p$-module homomorphism. We shall also need to know that there is a monomorphism of rings $\iota : R_p \to \hat{R}_p$, where $R_p := \{u/v : \ u, v \in R, \ v \text{ prime to } p\}$. By definition $\iota(u/v)$ is the element in $\lim_{\leftarrow} R/(p^n)$ with the component in $R/(p^n)$ equal to the class of u/v in $R/(p^n)$. (Since v is not divisible by p, the class $(v)_n$ is invertible in $R/(p^n)$, hence $(u/v)_n := (u)_n(v)_n^{-1}$ is meaningful.) We will thus regard R_p as a subring in $\hat{R}_p$.

3.2. **Torsion-free modules.**

Lemma 3.2. *Let V be torsion free. Then in the decomposition of V^* into a direct sum of indecomposable injective modules at least one summand must be K, hence the decomposition takes the form*

$$V^* = K \oplus (\oplus_{p \in P_V} R(p^\infty)), \tag{3.2}$$

where P_V is the set of all primes $p \in R$ such that $\mathrm{ann}_{V^*}(p) \neq 0$.

Proof. Otherwise V^* would be a torsion module, hence such would also be every quotient of V^*. But, since V is torsion-free, it contains a copy of R, hence R^* is a quotient of V^*. However, by Example 3.1 R^* is not torsion. □

With respect to the decomposition (3.2) each endomorphism of V^* is represented by a matrix of homomorphisms between various summands. An element of the center Z must commute in particular with the projections onto the summands, so is represented by a diagonal matrix b with the diagonal entries in K or in $\hat{R}_p$. If $r_0 \in K$ and $f \in \hat{R}_p$ are such entries, then

$$fg(r) = g(r_0 r) \quad (r \in K) \tag{3.3}$$

for each R-module homomorphism $g : K \to R(p^\infty)$, since g can be extended to the endomorphism of V^* by 0 on other positions in the corresponding matrix, and b must commute with this extension.

Lemma 3.3. *For a prime $p \in R$ the presence of the direct summand $R(p^\infty)$ in the decomposition (3.2) of V^* implies the equality $r_0 = f$ in $\hat{R}_p$. (More precisely, f must be equal to the image of r_0 in $\hat{R}_p$ under the ring monomorphism $R_p \to \hat{R}_p$.) This equality is also sufficient for (3.3).*

Proof. Let g be the composition of the quotient map $K \to K/R$ followed by the projection of K/R onto its p-primary component. The effect of g on any rational function r can be seen by expanding r into partial fractions with denominators powers of primes $q \in R$: g annihilates all the terms with denominators of the form q^n, $q \neq p$ $(n = 1, 2, \ldots)$, and leaves the terms with denominators of the form p^n unchanged, so the kernel of g is $K \cap \hat{R}_p$.

Set $r = 1$ in (3.3). Since $g(1) = 0$, it follows that $g(r_0) = 0$, which means that $r_0 \in K \cap \hat{R}_p$. If we show that

$$g(r_0 r) = r_0 g(r) \quad (r \in K), \tag{3.4}$$

then from (3.3) and (3.4) we will have $fg(r) = r_0 g(r)$, hence (since g is surjective) $f = r_0$. Let $r_0 = u/v$, where $u, v \in R$ are relatively prime; then r_0 is in $\hat{R}_p$ if and only if v is not divisible by p, and then v is invertible in $\hat{R}_p$. For any R-module map $g : K \to R(p^\infty)$ we have $vg(r_0 r) = g(v r_0 r) = g(ur) = ug(r)$, hence $g(r_0 r) = (u/v)g(r) = r_0 g(r)$.

Conversely, if $r_0 = f$ in $\hat{R}_p$, then since (3.4) holds for all module homomorphisms $g : K \to R(p^\infty)$, (3.3) also holds. □

Observe that the summand $R(p^\infty)$ is present in the decomposition of V^* if and only if $\ker p(a^*) \neq 0$, which is equivalent to $p(a)V \neq V$. Note also that, there are no nonzero R-module homomorphisms from $R(p^\infty)$ to K since $R(p^\infty)$ is torsion, while K is torsion-free. Thus, we may summarize the discussion so far in the following proposition.

Proposition 3.4. *If V is a torsion-free module over $R = \mathbb{F}[t]$, then the center of $\mathrm{L}_R(V)$ is isomorphic to the algebra $A := \cap_{p \in P_V}(K \cap \hat{R}_p)$, where $K = \mathbb{F}[t]$ and P_V is the set of all primes $p \in R$ such that $pV \neq V$. In other words, if an operator $a \in \mathrm{L}(V)$ is such that $p(a)$ is injective for all $p \in R$, then its bicommutant $(a)''$ is isomorphic to the subalgebra A of K consisting of all $r = u/v$ $(u, v \in R$ relatively prime$)$ such that $v(a)$ is invertible.*

3.3. **Torsion modules.** The center Z of $\mathrm{L}_R(V)$ for a torsion module V is already known [13]. We will now describe the result in a form needed later. Let P be the set of all primes in R and for each $p \in P$ let $V_p = \{\xi \in V : p^k(a)\xi = 0 \text{ for some } k \in \mathbb{N}\}$, the *p-primary part* of V. It is well-known [13] that

$$V = \oplus_{p \in Q_V} V_p, \tag{3.5}$$

where Q_V is the set of all $p \in P$ such that $V_p \neq 0$. Let us first consider the case when there is only one summand in this decomposition, that is $V = V_p$ for some $p \in P$. In this case V is equal to the union of the increasing sequence of submodules $U_n := \ker p^n(a)$. If $V = U_n$ for some n then a is algebraic and therefore each $b \in (a)''$ is of the form $b = f(a)$ for some polynomial p by [13, Exercise 90, p. 72]. In this case Z is isomorphic to $R/(p^n)$. In the remaining case, when $U_n \neq V$ for all n, Z is isomorphic to the ring $\hat{R}_p$ of all power series of the form (3.1).

Definition 3.5. For each series f of the form (3.1) define $f(a)$ as follows. Each $\xi \in V$ is in some U_m. Choose a polynomial $F_m \in R$ such that the coset of F_m in $R/(p^m)$ is equal to the image of f under the map $\hat{R}_p = \lim_{\leftarrow} R/(p^n) \to R/(p^m)$ and then let

$$f(a)\xi := F_m(a)\xi.$$

(We may simply take $F_m = \sum_{j=0}^{m-1} f_j p^j$, where the $f_j \in R$ are as in (3.1).)

If G_m is another such polynomial, then $G_m - F_m$ is divisible by p^m, hence $G_m(a)\xi = F_m(a)\xi$. Further, if we enlarge m to $m+1$, then F_{m+1} and F_m have the same coset in $R/(p^m)$, hence again $F_{m+1}(a)\xi = F_m(a)\xi$. Thus $f(a)$ is well defined. Since each U_m is invariant under $(a)'$, clearly $f(a) \in (a)''$.

It follows from known results (see [13, Theorem 29] or [15, Proof of Theorem 19.7]) that every $b \in (a)''$ is of the form $f(a)$ for an f as in (3.1). Since all the proofs of this which we have found in the literature require a more extensive knowledge of the structure theory of torsion modules than it is absolutely necessary, we will sketch now a more direct argument.

Clearly $\mathrm{L}_R(V) = \lim_{\leftarrow} \mathrm{L}_R(U_n)$, since each sequence of endomorphisms $g_n \in \mathrm{L}_R(U_n)$ satisfying $g_{n+1}|U_n = g_n$ defines an endomorphism $g \in \mathrm{L}_R(V)$. If we prove that every endomorphism b_n of the L_R-module U_n extends to an endomorphism b_{n+1} of U_{n+1}, then it follows easily that the center of $\mathrm{L}_R(V)$ is the inverse limit of the centers of $\mathrm{L}_R(U_n)$. These centers are isomorphic to $R/(p^nR)$ since $a|U_n$ is algebraic with the minimal polynomial p^n. (This is [13, Exercise 88]; perhaps the simplest proof is by using the fact that $R/(p^n)$ is a self-injective ring by [16, Corollary 3.13].) Now U_{n+1} is a torsion module with $p^{n+1}U_{n+1} = 0$, hence by Prüfer's theorem (which can again be seen as a consequence of self-injectivity by [16, Exercise 18, p. 115]) U_{n+1} is a direct sum of cyclic modules. The endomorphism rings of such modules can be described quite explicitly. For example, if U_{n+1} is cyclic, then so is U_n and all endomorphisms of U_{n+1} and U_n are polynomials in a. In general, endomorphism of U_{n+1} are suitable matrices whose entries are polynomials in a. By a closer examination of this structure it can be verified that each endomorphism of U_n extends to U_{n+1}.

For a torsion module V with the primary decomposition (3.5) the center of $\mathrm{L}_R(V)$ is just the product of the centers of $\mathrm{L}_R(V_p)$.

3.4. **Mixed modules.** To describe the center of $\mathrm{L}_R(V)$ for a general R-module V, let T be the torsion submodule of V and $W = V/T$. Assume that $W \neq 0$. Since W

is torsion-free, W^* is divisible, hence $V^* \cong T^* \oplus W^*$. Each central endomorphism b of V^* commutes in particular with the projections of V^* onto the two summands T^* and W^*, therefore it must be of the form

$$b = f \oplus r$$

for some endomorphisms f and r of T^* and W^* (respectively). By Proposition 3.4 r is essentially a rational function. Let $T = \oplus_{q\in Q_T} T_q$ be the decomposition of T into its primary summands, and let $f_q \in \mathrm{Z}(\mathrm{L}(T_q))$ be the components of f. Let $g_p \in \mathrm{L}(T^*, R(p^\infty))_R$ and $g_0 \in \mathrm{L}(T^*, K)_R$ be the components of g, and denote $g_{p,q} = g_p|T_q^*$. Then

$$rg_{p,q} = g_{p,q}f_q \quad \text{for all } g_{p,q} \in \mathrm{L}(T_q^*, R(p^\infty))_R \text{ or } g_{0,q} \in \mathrm{L}(T_q^*, K)_R. \tag{3.6}$$

We now study the question:

For which $q \in Q_T$ is it possible that $f_q \neq r$ in $\hat{R}_q$ (or in $R/(q^{n(q)})$) in spite of the condition (3.6)?

Lemma 3.6. *Let $r \in K$ and $\phi \in \hat{R}_q$. If T_q contains a nonzero divisible submodule or, if some nonzero quotient module of T_q is divisible, then the condition*

$$r\psi = \psi\phi \quad \text{for all } \psi \in \mathrm{L}(T^*, K)_R$$

implies that $r = \phi$ in $\hat{R}_q$.

The same conclusion also holds if T_q is a direct sum of cyclic modules, $T_q = \oplus_{j\in J} R/(q^{n(j)})$, such that the set $N_q := \{n(j) : j \in J\}$ is not bounded.

Proof. If T_q contains a nonzero divisible submodule, then it contains an indecomposable, necessarily q-torsion, such module, hence $R(q^\infty)$. For each n choose an isomorphism of R-modules from $(R/(q^n))^*$ to $R/(q^n)$, and therefore an isomorphism θ from $(R(q^\infty))^* = \lim_{\leftarrow}(R/(q^n))^*$ onto $\lim_{\leftarrow} R/(q^n) = \hat{R}_q$. Observe that θ is then also a homomorphism of $\hat{R}_q$ modules. Let $\pi : T_q^* \to \hat{R}_q$ be the composition $\pi = \theta\pi_0$, where $\pi_0 : T_q^* \to (R(q^\infty))^*$ is the quotient map (the adjoint of the inclusion $R(q^\infty) \to T_q$). Since K is injective, the inclusion $K \cap \hat{R}_q \to K$ can be extended to an R-module homomorphism $h : \hat{R}_q \to K$. Let $\psi = h\pi$. Then, from the condition $r\psi = \psi\phi$ we compute for every $\xi \in T_q^*$, since π is an $\hat{R}$-module homomorphism,

$$rh\pi(\xi) = r\psi(\xi) = \psi(\phi\xi) = h\pi(\phi\xi) = h(\phi\pi(\xi)).$$

Since π is surjective, this implies that

$$rh(\eta) = h(\phi\eta) \quad \text{for all } \eta \in \hat{R}_q.$$

If we take $\eta = 1$, we get (since $h|K \cap \hat{R}_q$ acts as the identity)

$$r = h(\phi). \tag{3.7}$$

If we can show that $\phi \in K \cap R_q$, then $h(\phi) = \phi$ and we will have $r = \phi$ as claimed. Suppose that $\phi \notin K \cap R_q$, that is $\phi \notin K$. Thus $z\phi \neq s$ for all nonzero $z \in R$ and $s \in K$, hence each element of $(K \cap \hat{R}_q) + R\phi$ can be uniquely expressed as $s + z\phi$ ($s \in K \cap \hat{R}_q$, $z \in R$). But then, for any $r_0 \in K$, we can first extend the inclusion $K \cap \hat{R}_q \to K$ to the R-module map $h_0 : (K \cap \hat{R}_q) + R\phi \to K$ by $h_0(s + z\phi) = s + zr_0$, and then further extend h_0 to an R-module map $h : \hat{R}_q \to K$. For such an h we have $h(\phi) = r_0$, which contradicts (3.7) if we choose $r_0 \neq r$.

If a nonzero quotient D of T_q is divisible, then $R(q^\infty)$ must be a quotient of T_q (since $R(q^\infty)$ is a direct summand, hence also a quotient, of D), hence $(R(q^\infty))^*$ is a submodule of T_q^*. With θ and h as in the previous paragraph, let now $\psi : T_q^* \to K$ be an R-module homomorphic extension of $h\theta$. Now the equality $r\psi(\xi) = \psi(\phi\xi)$ holds in particular for all $\xi \in (R(q^\infty))^*$ and, since θ is an isomorphism of $\hat{R}_q$-modules, this implies that $rh(\eta) = h(\phi\eta)$ for all $\eta \in \hat{R}_q$. The argument from the previous paragraph shows now that $r = \phi$ in $\hat{R}_q$.

If T_q is a direct sum of cyclic modules, as in the second part of the lemma, such that N_q is not bounded, choose a sequence $(j_k)_{k\in\mathbb{N}} \subseteq J$ such that $n(j_k) \geq k$ for all $k \in \mathbb{N}$. Then there exist $\hat{R}_q$-module monomorphisms $\iota_k : R/(q^k) \to R/(q^{n(j_k)})$, which induce an embedding $\iota : \prod_{k\in\mathbb{N}} R/(q^k) \to \prod_{k\in\mathbb{N}} R/(q^{n(j_k)})$. There is also a natural embedding $\kappa : \hat{R}_q \to \prod_{k\in\mathbb{N}} R/(q^k)$ of $\hat{R}_q$-modules, given by

$$\kappa(\sum_{i=0}^{\infty} c_i q^i) = ([c_0], [c_0 + c_1 q], \ldots, [\sum_{i=0}^{n} c_i q^i], \ldots) \quad (c_i \in R,\ \operatorname{degree} c_i < \operatorname{degree} q).$$

Finally, since $(j_k)_{k\in\mathbb{N}}$ is a subset of J, we have the obvious embedding

$$\tau : \prod_{k\in\mathbb{N}} R/(q^{n(j_k)}) \to \prod_{j\in J} R/(q^{n(j)}).$$

Let $\sigma : \hat{R}_q \to T_q^*$ be the composition $\sigma = \theta^{-1}\tau\iota\kappa$, where

$$\theta : T_q^* = \prod_{j\in J} (R/(q^{n(j)})^* \to \prod_{j\in J} R/(q^{n(j)}$$

is the isomorphism defined via some isomorphisms $(R/(q^{n(j)})^* \to R/(q^{n(j)})$ of $\hat{R}_q$-modules. Now, since all these are $\hat{R}_q$-module maps, we may regard $\hat{R}_q$ as a submodule in T_q^* and the proof can be completed as before, using the map h. □

If T_q does not have any nonzero divisible submodule, then there exists in T_q a submodule B_q such that B_q is a direct sum of cyclic modules and T_q/B_q is divisible. (This follows from Kulikov's theorem [18, 4.3.4] for $\mathbb{Z}$-modules, the proof for $\mathbb{F}[t]$-modules is essentially the same.) So, unless $T_q = B_q$, T_q has a nonzero divisible quotient and therefore by Lemma 3.6 the condition (3.6) implies that $f_q = r$. In the remaining case, $T_q = B_q$, T_q is a direct sum of cyclic modules as in the second part of Lemma 3.6, hence, if the set N_q is not bounded (3.6) again implies that $f_q = r$. On the other hand, if the set N_q is bounded, say by the least upper bound $n(q)$, then $q^{n(q)}T_q = 0$ implies that T_q^* is a torsion module, hence there can be no nonzero module maps from T_q^* to K. But there are nonzero such maps from T_q^* to $R(p^\infty)$ if $p = q$ for some $p \in P_W$, where P_W is the set of all primes $p \in R$ such that W^* contains $R(p^\infty)$. Namely, in this case $R(q^\infty)$ and T_q^* each contains its own copy of $R/(q^{n(q)})$ and if we choose $g_{q,q}$ so that it identifies these two copies isomorphically, then we see easily that (3.6) implies that $f_q = r$ modulo $(q^{n(q)})$. This proves:

Lemma 3.7. *If $f_q \neq r$, then q is necessarily in the subset S of $Q_T \setminus P_W$ consisting of those q for which T_q is a direct sum of cyclic modules of bounded orders $\leq n_q$.*

Finally, let $S_b = \{q \in S :\ f_q \neq r \text{ in } R/(q^{n(q)})\}$.

Lemma 3.8. *The set S_b is finite.*

Proof. If not, then there exists a sequence $S_1 = \{q_0, q_1, \ldots\} \subseteq S_b$ such that $q_j \neq q_i$ if $j \neq i$. Consider the module

$$(\bigoplus_{j\in\mathbb{N}} R/(q_j^{n_j}))^* \cong \prod_{j\in\mathbb{N}} (R/(q_j^{n_j}))^* =: U,$$

which is a direct summand in T^*. Let $r = u/v$ with u, v relatively prime polynomials. Let $\omega \in U$ be defined by $\omega = (\omega_j)_{j\in\mathbb{N}}$, where $\omega_j \in (R/(q_j^{n_j}))^*$ is such that $q_j^{n_j-1}\omega_j \neq 0$, and let $\rho = (\rho_j)_{j\in\mathbb{N}}$ be $\rho = v\omega$. For any $g_0 \in \mathrm{L}(T^*, K)_R$ we have that $rg_0(\rho) = g_0(f\rho)$ (since $(f \oplus r) \in Z$), hence $ug_0(\rho) = vg_0(f\rho)$, consequently $vg_0(u\omega) = vg_0(fv\omega)$, thus

$$g_0(u\omega - vf\omega) = 0.$$

Since this holds for all g_0 and K is injective, this implies that the element $u\omega - vf\omega$ is torsion, say $h(u-vf)\omega = 0$ for a polynomial h. Since the components of this element are $h(u - vf_{q_j})\omega_j$, it follows from the definition of ω that $q_j^{n_j}$ divides $h(u - vf_{q_j})$. Since h has only finitely many divisors, it follows that for all sufficiently large j $u - vf_{q_j}$ must be divisible by $q_j^{n_j}$, say $u - vf_{q_j} = s_j q_j^{n_j}$ for a polynomial s_j. Then

$$\frac{u}{v} = f_{q_j} + \frac{s_j}{v} q_j^{n_j}. \tag{3.8}$$

Since v can be divisible only by finitely many prime factors q_j, $1/v$ acts as a multiplication by a polynomial on $(R/(q_j^{n_j}))^*$ if j is large enough. (If $1 = mv + zq^{n_j}$ for some polynomials m, z, then $1/v$ acts as the multiplication by m.) Thus (3.8) implies that r and f_{q_j} coincide as operators on $(R/(q_j^{n_j}))^*$ for large enough j, hence also on $R/(q_j^{n_j}) \cong (R/(q_j^{n_j}))^*$, but this contradicts the definition of S_b. $\square$

Observe that the submodule $T_b := \oplus_{q\in S_b} T_q$ of V is *pure* in the following sense: given $\eta \in T_b$ and $p \in R$, if there exists $\xi \in V$ such that $p\xi = \eta$, then there exists $\zeta \in T_b$ such that $p\zeta = \eta$. Since S_b is finite, we can form $q_b = \prod_{q\in S_b} q^{n_q}$ and clearly $q_b T_b = 0$. It follows now from [13, Theorem 7] that T_b is a direct summand in V, say $V = T_b \oplus W_b$. Let $T_0 = \oplus_{q\in Q_T\setminus S_b} T_q$, so that $T = T_b \oplus T_0$. Then

$$T_b^* \oplus W_b^* = V^* = T^* \oplus W^* = T_b^* \oplus T_0^* \oplus W^*,$$

hence W_b^* is naturally isomorphic to $T_0^* \oplus W^*$ (both are isomorphic to V^*/T_b^*). Since b acts on $T_0^* \oplus W^*$ as the multiplication by r (by the definition of S_b), the same must hold for the action of b on W_b^* and consequently also on W_b.

The above discussion and lemmas prove the following theorem in the harder direction.

Theorem 3.9. *Let T be the torsion submodule of a module V over $R = \mathbb{F}[t]$, $W = V/T$ and Z the center of $\mathrm{L}_R(V)$. Denote by P_W the set of all primes $p \in R$ such that $pW \neq W$, by Q_T the set of all primes $q \in R$ such that the q-primary part T_q of T is nonzero and by S the set of all $q \in Q_T \setminus P_W$ such that T_q is a direct sum of cyclic modules of orders bounded by n_q (that is, $q^{n_q}T_q = 0$ for $n_q \in \mathbb{N}$ and we choose the minimal such n_q). Suppose that $W \neq 0$. Then for each $b \in Z$ there exist a finite subset S_b of S and a submodule W_b of V such that*

$$V = T_b \oplus W_b, \quad \textit{where} \quad T_b = \oplus_{q\in S_b} T_q,$$

b acts on W_b as the multiplication by a rational function $r \in K$ and b acts on each summand T_q of T_b as the multiplication by a polynomial f_q. Conversely, any map b on V for which there exist such a decomposition of V and b is in Z.

Proof. It only remains to prove the sufficiency of the stated conditions for b to be in Z. So, let $b = (\oplus_{q\in S_b} f_q) \oplus r$ be the decomposition of b with respect to the decomposition

$$V = (\oplus_{q\in S_b} T_q) \bigoplus W_b \tag{3.9}$$

of V, where $r \in K$ and $f_q \in R/(q^{n_q})$. Because of incompatible torsion there can be no nonzero module homomorphisms between T_{q_1} and T_{q_2} for different q_1, q_2 in S_b. If we can show that there are no nonzero homomorphisms from T_q to W_b and from W_b to T_q, then each endomorphism of V will be represented by a diagonal matrix relative to the decomposition (3.9), hence clearly b will be in Z.

Note that for $q \in S_b$ the multiplication by q acts as an invertible operator on $W_b^* = T_0^* \oplus W^*$ (hence also invertible on W_b) since it acts as an invertible operator on each primary summand T_p of T_0 (hence also on the dual T_0^* of the direct sum of such summands) and also on each summand $R(p^\infty)$ of W^* (since $q \notin P_W$). Since $q^{n_q}T_q = 0$, while the multiplication by q acts as an invertible operator on W_b, $\mathrm{L}_R(T_q, W_b) = 0$. To show that also $\mathrm{L}_R(W_b, T_q) = 0$, it suffices to show that $\mathrm{L}(T_q^*, W_b^*) = 0$. But this follows from $q^{n_q}T_q^* = 0$ and the fact that q acts as an invertible operator on W_b^*. □

Example 3.10. If V contains a copy of R as a direct summand, then $\mathrm{Z}(\mathrm{L}_R(V)) = R$. This is known (see [13, Exercise 95]), but also readily follows from Theorem 3.9. Namely, let $V = R \oplus V_0$. The torsion submodule T of V is the torsion submodule of V_0, and $W := V/T = R \oplus (V_0/T)$. Hence $pW \neq W$ for all primes $p \in R$, so P_W is the set of all primes in R and $S \subseteq Q_T \setminus P_W = \emptyset$. Thus $Z \subseteq K$. Since the denominators of functions in Z are not divisible by any prime $p \in R$ (because P_W contains all primes in R), Z must be R.

Similarly, if V contains a copy R_0 of R such that the quotient R-module V/R_0 is torsion-free, then $\mathrm{Z}(\mathrm{L}_R(V)) = R$. In this case V is necessarily torsion free, and it is easy to see that $pV \neq V$ for all primes $p \in V$.

4. The range inclusion for derivations

Let $\mathrm{F}(V)$ be the vector space of all finite rank linear operators on a vector space V. $\mathrm{F}(V)$ is naturally isomorphic to $V \otimes V^*$ by the isomorphism which sends $\xi \otimes \rho$ ($\xi \in V$, $\rho \in V^*$) to the rank 1 operator $\eta \mapsto \rho(\eta)\xi$. Thus the dual space of $\mathrm{F}(V)$ can be identified with $(V \otimes V^*)^* = \mathrm{L}(V^*)$.

Theorem 4.1. *Let d_a and d_b be the derivations on $\mathrm{L}(V)$ induced by operators $a, b \in \mathrm{L}(V)$. Then $\ker d_a \subseteq \ker d_b$ if and only if $d_b(\mathrm{F}(V)) \subseteq d_a(\mathrm{F}(V))$. In this case the inclusion $\mathrm{im}\, d_{b^*} \subseteq \mathrm{im}\, d_{a^*}$ also holds.*

Proof. Note that $\ker d_a$ is just the commutant $(a)'$ of a and that the inclusion $(a)' \subseteq (b)'$ is equivalent to $b \in (a)''$. (Indeed, by taking the commutants we infer from $(a)' \subseteq (b)'$ that $b \in (a)''$. Conversely, $b \in (a)''$ implies that $(a)''' \subseteq (b)'$; but $A''' = A'$ for any subalgebra A of $\mathrm{L}(V)$, as it is easy to deduce from the obvious inclusion $A \subseteq A''$.) Thus the conditions $\ker d_a \subseteq \ker d_b$ and $b \in (a)''$ are equivalent. By the same argument the conditions $\ker d_{a^*} \subseteq \ker d_{b^*}$ and $b^* \in (a^*)''$ are also equivalent. But by Theorem 2.4 $b \in (a)''$ if and only if $b^* \in (a^*)''$, hence it follows that the two conditions $\ker d_a \subseteq \ker d_b$ and $\ker d_{a^*} \subseteq \ker d_{b^*}$ are equivalent. It is easy to verify that $d_{a^*} = -(d_a)^*$, hence $\ker d_{a^*} = (d_a(\mathrm{F}(V)))^\perp$ and similarly for b, so the last inclusion is equivalent to $d_b(\mathrm{F}(V)) \subseteq d_a(\mathrm{F}(V))$.

If $\ker d_a \subseteq \ker d_b$, then

$$\operatorname{im} d_{b^*} = \operatorname{im}((d_b|\mathrm{F}(V))^*) = (\ker d_b|\mathrm{F}(V))^\perp \subseteq (\ker d_a|\mathrm{F}(V))^\perp = \operatorname{im}((d_a|\mathrm{F}(V))^*) = \operatorname{im} d_{a^*}.$$

□

Is there any connection between the two range inclusions

$$d_b(\mathrm{F}(V)) \subseteq d_a(\mathrm{F}(V)) \tag{4.1}$$

and

$$d_b(\mathrm{L}(V)) \subseteq d_a(\mathrm{L}(V))? \tag{4.2}$$

The following example show that (4.2) does not imply (4.1).

Example 4.2. Let $V = \mathbb{F}^{(\mathbb{N})}$ and $a \in \mathrm{L}(V)$ be as in Example 3.1, so that any $b \in (a)''$ is a polynomial in a. We may represent operators in $\mathrm{L}(V)$ by $\mathbb{N} \times \mathbb{N}$ matrices (that have in each column only finitely many nonzero elements). A short computation shows that d_a is surjective on $\mathrm{L}(V)$. Thus the inclusion (4.2) holds for all $b \in \mathrm{L}(V)$, not just for $b \in (a)''$. (In this respect a behaves quite differently from the shift operator on the Hilbert space ℓ_2; for the latter see [22].)

In the rest of the paper we will try to prove that (4.1) implies (4.2), which is (by the proof of Theorem 4.1) equivalent to the following theorem.

Theorem 4.3. *Let $a, b \in \mathrm{L}(V)$. If $b \in (a)''$, then the range inclusion $d_b(\mathrm{L}(V)) \subseteq d_a(\mathrm{L}(V))$ holds.*

We will divide the proof of this theorem into several cases.

4.1. **The case when $(a)''$ is isomorphic to a subalgebra of $\mathbb{F}[t]$.** First we will consider the simple case when the center of $\mathrm{L}_R(V)$ is just $R = \mathbb{F}[t]$ (hence any $b \in (a)''$ is a polynomial in a). For this and a later use, it will be convenient to have the following definition.

Definition 4.4. For a polynomial $p(t) = \sum_{j=0}^n \alpha_j t^j$ the derivative of the map $a \mapsto p(a)$ at a point $a \in \mathrm{L}(V)$ is the linear map $\dot{p}_a : \mathrm{L}(V) \to \mathrm{L}(V)$ defined by

$$\dot{p}_a(x) = \sum_{j=1}^n \alpha_j \sum_{i=0}^{j-1} a^{j-i-1} x a^i \quad (x \in \mathrm{L}(V)).$$

Clearly $\dot{p}_a$ is an $(a)'$-bimodule endomorphism of $\mathrm{L}(V)$ and a simple computation shows that

$$d_a(\dot{p}_a(x)) = d_{p(a)}(x) \tag{4.3}$$

for all $x \in \mathrm{L}(V)$. Thus $\operatorname{im} d_{p(a)} \subseteq \operatorname{im} d_a$. Moreover, the following form of Leibnitz rule can also be easily verified:

$$(pq)\dot{}_a(x) = \dot{p}_a(x)q(a) + p(a)\dot{q}_a(x) \quad (x \in \mathrm{L}(V),\ p, q \in R). \tag{4.4}$$

This suggest us to define $\dot{r}_a(x)$ for any rational function $r = p/q$, such that $q(a)$ is invertible, to be the unique operator satisfying the following two equivalent identities:

$$q(a)\dot{r}_a(x) + \dot{q}_a(x)r(a) = \dot{p}_a(x) = \dot{r}_a(x)q(a) + r(a)\dot{q}_a(x); \tag{4.5}$$

that is

$$\dot{r}_a(x) = q(a)^{-1}\dot{p}_a(x) - q(a)^{-1}\dot{q}_a(x)r(a) \quad (x \in \mathrm{L}(V)).$$

Then it can be proved that (4.3) and (4.4) hold for rational functions. (To prove that $(rs)\dot{}_a(x) = \dot{r}_a(x)s(a) + r(a)\dot{s}_a(x)$ for two rational functions r and s, one shows that $\dot{r}_a(x)s(a) + r(a)\dot{s}_a(x)$ satisfies one of the two defining identities (4.5) for the product rs in place of r.) Then (4.3) implies that the range inclusion $\operatorname{im} d_b \subseteq \operatorname{im} d_a$ holds whenever the center of $\mathrm{L}_R(V)$ is a subalgebra of $K = \mathbb{F}(t)$, in particular if V is torsion-free, by Proposition 3.4.

4.2. **The case of locally algebraic operators.** We study next the case of torsion modules, that is, we assume that $a \in \mathrm{L}(V)$ is locally algebraic. If a is algebraic, each $b \in (a)''$ is of the form $b = f(a)$ for a polynomial f, and $d_b = d_a\dot{f}_a$ by (4.3), hence $\operatorname{im} d_b \subseteq \operatorname{im} d_a$. Assume now that a is not algebraic but that V is p-primary for some prime $p \in R$, that is, $V = \cup_{m=1}^{\infty} U_m$, where $U_m = \ker p(a)^m$. Then $b = f(a)$ for a power series of the form (3.1) (see Subsection 3.3). For such a series f we would like to define $\dot{f}_a : \mathrm{L}(V) \to \mathrm{L}(V)$.

Definition 4.5. Given $\xi \in V$, choose m so that $\xi \in U_m$, and then choose k such that all the vectors $\dot{p}_a(x)p(a)^j\xi$ $(j = 0, \ldots m)$ are in U_k and set $n = k + m$. Choose a polynomial $F_n \in R$ such that the coset of F_n in $R/(p^n)$ is equal to the image of f under the natural map $\hat{R}_p \to R/(p^n)$ (for example $F_n = \sum_{j=0}^{n-1} f_j p^j$, where f_j are as in (3.1)) and set

$$\dot{f}_a(x)\xi = (F_n)\dot{}_a(x)\xi. \tag{4.6}$$

We have to show that this definition is independent of the choices of m, k and F_n. If G_n is another such polynomial, then $G_n - F_n = qp^n$ for a polynomial $q \in R$. Using the product formula (4.4) and induction we have (since $p(a)^n\xi = 0$) that

$$(qp^n)\dot{}_a(x)\xi = \dot{q}_a(x)p^n(a)\xi + q(a)(p^n)\dot{}_a(x)\xi = q(a)\sum_{j=0}^{n} p(a)^{n-j-1}\dot{p}_a(x)p(a)^j\xi = 0,$$

since $n-j-1 \geq k$ for all $j = 0, \ldots, m-1$. Thus $(G_n)\dot{}_a(x)\xi = (F_n)\dot{}_a(x)\xi$. Enlarging m and k would give us a polynomial congruent to F_n module p^n. This shows that $\dot{f}_a$ is well defined. Moreover, it can be readily verified that $\dot{f}_a$ is an $(a)'$-bimodule map on $\mathrm{L}(V)$, and that $d_a\dot{f}_a = d_{f(a)} = d_b$, hence $\operatorname{im} d_b \subseteq \operatorname{im} d_a$.

In the general case of several primary summands, relative to the decomposition $V = \oplus_{p\in P} V_p$ each operator $x \in \mathrm{L}(V)$ is represented by an infinite operator matrix $[x_{p,q}]$, where $x_{p,q}$ is an operator from V_q to V_p. Denoting by a_p and b_p the restrictions of a and b to V_p, the proof that $\operatorname{im} d_b \subseteq \operatorname{im} d_a$ reduces to showing that the equation

$$a_p x - x a_q = y \tag{4.7}$$

has a solution x in $\mathrm{L}(V_q, V_p)$ for each $y \in \mathrm{L}(V_q, V_p)$ of the form $b_p z - z b_q$. If $q = p$, this has just been proved in the previous paragraph, while if $q \neq p$ it is a consequence of Lemma 4.6 below. We remark that the equation of the form $cx - xd = y$ has been studied in the analytic context using the notion of spectrum (see e.g. [19] or [17, p. 8]), but this method does not apply to the purely algebraic context of Lemma 4.6.

Lemma 4.6. *Let c, e be linear operators on vector spaces V_c and V_e. If there exists a polynomial v such that $v(e) = 0$ and $v(c)$ is invertible, then for each $y \in \mathrm{L}(V_e, V_c)$ the equation*

$$cx - xe = y \tag{4.8}$$

has a unique solution $x \in \mathrm{L}(V_e, V_c)$.

The same conclusion holds under the assumption that one of the spaces V_c, V_e, say V_e, is of the form $V_e = \cup_{n=1}^{\infty} \ker q(e)^n$ for a polynomial q such that $q(c)$ is invertible.

Proof. Observe that for any polynomial $f = \sum \alpha_j t^j$, $x \in \mathrm{L}(V_e, V_c)$ and $y := cx - xe$ the following identity holds:

$$f(c)x - xf(e) = \sum_j \alpha_j \sum_{i=0}^{j-1} c^{j-i-1} y e^i =: \dot{f}_{c,e}(y) \tag{4.9}$$

Applying this to $f = v$, if $v(e) = 0$ and $v(c)$ is invertible, we deduce that

$$x = v(c)^{-1} \dot{v}_{c,e}(y). \tag{4.10}$$

Conversely, a direct computation (using the definition (4.9) of $\dot{f}_{c,e}(y)$) verifies that x given by (4.10) is indeed a solution of (4.8).

In a more general situation of the second part of the lemma, we try to solve the equation (4.8) locally, that is, for each $\xi \in V_e$ we choose n so that $\xi \in \ker q(e)^n$ and then, as suggested by (4.10) we set

$$x\xi := q(c)^{-n} (q^n)\dot{}_{c,e}(y)\xi,$$

To show that $x\xi$ is independent of the choice of n, note (by a straightforward computation) for any two polynomials u, v the equality

$$(uv)\dot{}_{c,e}(y) = \dot{u}_{c,e}(y)v(e) + u(c)\dot{v}_{c,e}(y).$$

Applying this to $u = q$ and $v = q^n$, we obtain (since $q(e)^n \xi = 0$) that

$$q(c)^{-n-1} (q^{n+1})\dot{}_{c,e}(y)\xi = q(c)^{-n} (q^n)\dot{}_{c,e}(y)\xi.$$

So x is a well-defined map and it is easy to verify that x is linear and satisfies the equation (4.8). $\square$

4.3. **The general non-torsion case.** For a general non-torsion module V we know from Theorem 3.9 that for every b in $(a)''$ (= the center of $\mathrm{L}_R(V)$) V decomposes into a direct sum $V = (\oplus_{q \in S_b} T_q) \oplus W_b$ so that b acts on W_b as the multiplication by a rational function r (that is, $b|W_b = r(a|W_b)$) and b acts on each T_q as the multiplication by a polynomial f_q. Moreover, $q(a)|T_p$ is invertible if $p, q \in S_b$ are different (since p and q are different primes in R). Also $q(a)|W_b$ is invertible for $q \in S_b$, as we have seen in the proof of Theorem 3.9. If we now represent operators in $\mathrm{L}(V)$ by matrices relative to this decomposition of V, we may again use Lemma 4.6, in the same way as above, to show that the equation $ax - xa = y$ has a solution for each $y \in \operatorname{im} d_b$.

Remark 4.7. Let $b \in (a)''$. An examination of the above arguments shows that there exists an $(a)'$-bimodule endomorphism $D_{a,b} : \mathrm{L}(V) \to \mathrm{L}(V)$ such that $d_a D_{a,b} = d_b$. (For example, if $b = f(a)$ for a polynomial f, then $D_{a,b} := \dot{f}_a$.)

5. A counterexample among bounded operators

An example will show now that the results of Section 2 can not be generalized to bounded operators. In the example a Banach space X and operators $a, b \in \mathrm{B}(X)$ (the algebra of all bounded operators on X) will be presented such that b is in the bicommutant $(a)''$ of a in $\mathrm{B}(X)$, but nevertheless its adjoint $b^\sharp$ is not in the bicommutant of $a^\sharp$ in $\mathrm{B}(X^\sharp)$. Here we use the notation $X^\sharp$ for the Banach space dual of X, to distinguish it from the linear space dual X^*. Similarly, $a^\sharp$ denotes the bounded adjoint operator of a acting on $X^\sharp$.

Example 5.1. Let H be a separable infinite dimensional Hilbert space, B the algebra of all bounded linear operators on H, K the ideal in B of all compact operators, $C = B/K$ the Calkin algebra and T the ideal in B of all trace class operators. It is well-known (see e.g. [21, Section II.1] or [12]) that $K^\sharp = T$ and $B^\sharp = T \oplus K^\perp$, where $K^\perp$ is the annihilator of K in $B^\sharp$, and $K^\perp = C^\sharp$.

Let $a_0 \in K$ be an injective operator represented in some orthonormal basis of H by a diagonal matrix with the diagonal entries α_n, with $\alpha_n \neq \alpha_m$ if $n \neq m$. Denote by D the algebra of all operators in B that can be represented by diagonal matrices with respect to the same orthonormal basis as a_0. Thus D is the commutant and the bicommutant of a_0 in B. Finally, let $a \in \mathrm{B}(B)$ be the left multiplication by a_0 (that is, $a(x) = a_0 x$ for all $x \in B$). Note that $a^\sharp$ is the right multiplication by a_0 on $B^\sharp$ (where ρa_0 is defined by $\langle \rho a_0, x\rangle = \langle \rho, a_0 x\rangle$, for all $\rho \in B^\sharp$ and $x \in B$), T is invariant under $a^\sharp$, and $a^\sharp(C^\sharp) = 0$ since a_0 is compact. Thus $a^\sharp$ on $B^\sharp = T \oplus C^\sharp$ decomposes into the direct sum $a^\sharp = (a^\sharp|T) \oplus 0$, hence $(a^\sharp)'$ contains all maps on $T \oplus C^\sharp$ of the form $0 \oplus h$, where h is the right multiplication on $C^\sharp$ by any element $\dot{c} \in C$. We can choose $\dot{c} \in C$ and $d_0 \in D$ so that $\dot{c}\dot{d}_0 \neq \dot{d}_0\dot{c}$, where $\dot{d}_0$ denotes the coset of d_0 in the Calkin algebra C. (Indeed, since D is equal to its own commutant in B, the same holds for the image of D in C by [10].) Let $d : B \to B$ be the left multiplication by d_0. We will show that $d \in (a)''$ and that nevertheless $d^\sharp \notin (a^\sharp)''$.

Let $f \in (a)' \subseteq \mathrm{B}(B)$ and let $f = f_n + f_s$ be the decomposition of f into the normal part f_n and the singular part f_s (this means that f_n is weak* continuous and $f_s(K) = 0$, see [12, Chapter 10]). Then f_n and f_s are both in $(a)'$. (Indeed, the equality $fa = af$ can be written as $f_n a - af_n = af_s - f_s a$, where the left side is normal and the right side is singular, so they are both 0.) So $f_s a = af_s$, which means that

$$f_s(a_0 x) = a_0 f_s(x) \quad \text{for all } x \in B.$$

Since $a_0 \in K$, $f_s(a_0 x) = 0$, hence also $a_0 f_s(x) = 0$ for all $x \in B$. Since a_0 is injective, we deduce that $f_s = 0$, hence $f = f_n$ is normal. Let (a_n) be a sequence in the algebra generated by a_0, converging to d_0 in the weak* topology. Since $f \in (a)'$, f commutes with the left multiplications by all a_n. Then the weak* continuity of f implies that

$$f(d_0 x) = \lim_n f(a_n x) = \lim_n a_n f(x) = d_0 f(x) \quad \text{for all } x \in B.$$

Hence d commutes with f and therefore $d \in (a)''$.

If $d^\sharp$ were in $(a^\sharp)''$, then in particular $d^\sharp|C^\sharp$ would commute with the right multiplication h by $\dot{c}$ on $C^\sharp$, since $0 \oplus h$ is in $(a^\sharp)'$ by the second paragraph of this example. This would imply that $\dot{d}_0\dot{c} = \dot{c}\dot{d}_0$, a contradiction with the choice of $\dot{c}$ and $\dot{d}_0$.

There is no simple description known for the bicommutant of a general bounded operator $a \in \mathrm{B}(H)$, where H is a Hilbert or a Banach space. (There exist, however, representations of an operator algebra, say the one generated by a and 1, or a dual Banach algebra, such that its bicommutant is simply its weak* closure, see [1] and [5].) Further, the exact analogy of Theorem 4.3 in the context of $\mathrm{B}(H)$ is not true even for normal operators [11]. But perhaps a more proper formulation of the problem is as follows.

Problem. Suppose that operators $a, b \in \mathrm{B}(H)$ satisfy $\|d_b(x)\| \leq \kappa \|d_a(x)\|$ for all $x \in \mathrm{B}(H)$, where κ is a constant. Is then necessarily $d_b(\mathrm{B}(H)) \subseteq d_a(\mathrm{B}(H))$?

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